Conformal Field Theory Ground States as Critical Points of an Entropy Function

Author: Prashanth Shankar

[redacted]

September 2023

Abstract

We review the recent paper ”Conformal Field Theory Ground States as Critical Points of an Entropy Function”. We present the necessary background material in Conformal Field theory and Quantum Information. They include primary fields in CFT, Entanglement Hamiltonian of a ground state 1+1D CFT, and Entanglement entropy. We then discuss the main results of the paper, the conjectured entropy formula that is satisfied by the ground states of 1+1D conformal field theories. We show Entanglement Hamiltonian conjectured in the paper is unique up to a linear combination. We then conclude by discussing some of the questions the paper raises and further studies.

Acknowledgements

[redacted]

Contents

1 Introduction

Symmetry has played an important role in the understanding of the universe. Especially after the discovery of Noether's Theorem, we now understand these symmetries are more than mathematical beauty but also related to physical quantities, specifically conserved quantities. Conserved quantities lead to interesting laws of nature.

In Quantum Field Theory, fields and particles are described by field operators that transform under the representations of the Poincare group. A Theory that has the privilege of having Translation, rotation, and Boost symmetries. Knowing this, one can wonder if are there more symmetries in nature that can provide us more insight into the nature of the universe and provide us with more mathematical tools that can help us discover more.

Conformal Symmetry, the underlying symmetry of Conformal Field Theory, is a symmetry with scale invariance. This has helped us go beyond traditional QFT and explain things like critical phenomena in statistical physics [3], quantum Hall systems [6], and has applications in string theory [2] and more.

In a seemingly different field of information theory, Claude Shannon laid the mathematical foundation and established the relation between Information and Entropy [13]. Today the study of entropy is ubiquitous in Theoretical Physics. In research topics like the Black Hole Information Paradox [7], Holographic principle [14], Thermodynamics, and Cosmology understanding Entropy plays an important role.

In classical theory, understanding Entropy answers most of the questions we would have about uncertainty and information of the system. But that is unfortunately not enough for quantum systems. We will have to take into account Entanglement! There are very few things that are as mysterious as Entanglement. Fortunately, we have a good framework to measure and understand Entanglement. For this reason, Entanglement Entropy and Entanglement Hamiltonian will be of interest throughout our discussion.

Identifying 1+1D Conformal Field Theory ground state is useful in many research topics like String theory, Statistical Mechanics, and AdS/CFT Correspondence. In this report, we look at a formulation of Entanglement Entropy and Entanglement Hamiltonian functions whose critical points are satisfied by the ground state of 1+1D Conformal Field Theory.

We discuss the conditions for a state to be a critical point of the entropy function. Provide justification for the formula used and future speculate if more ambitiously are the defined criteria satisfied by only the ground state of 1+1D Conformal Field Theory.

2 Conformal Field Theory

2.1 Conformal Invarience

A conformal transformation is simply a transformation that transforms the metric this way [11]

$$g'_{\rho\sigma}(x')\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x)\, g_{\mu\nu}(x) \tag{2.1}$$

But what does this mean? It's a transformation of the coordinates that has an invertible mapping $x \to x'$ which leaves the metric invariant up to a scale. It leaves the angles invariant under these transformations. For flat space-time, the equation is simply [11]

$$\boxed{\eta_{\rho\sigma}(x')\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x)\, \eta_{\mu\nu}(x)} \tag{2.2}$$

Clearly, Poincare transformations are a subset of these as we can obtain the formula for Poincare transformation by setting $\Lambda(x) = 1$.

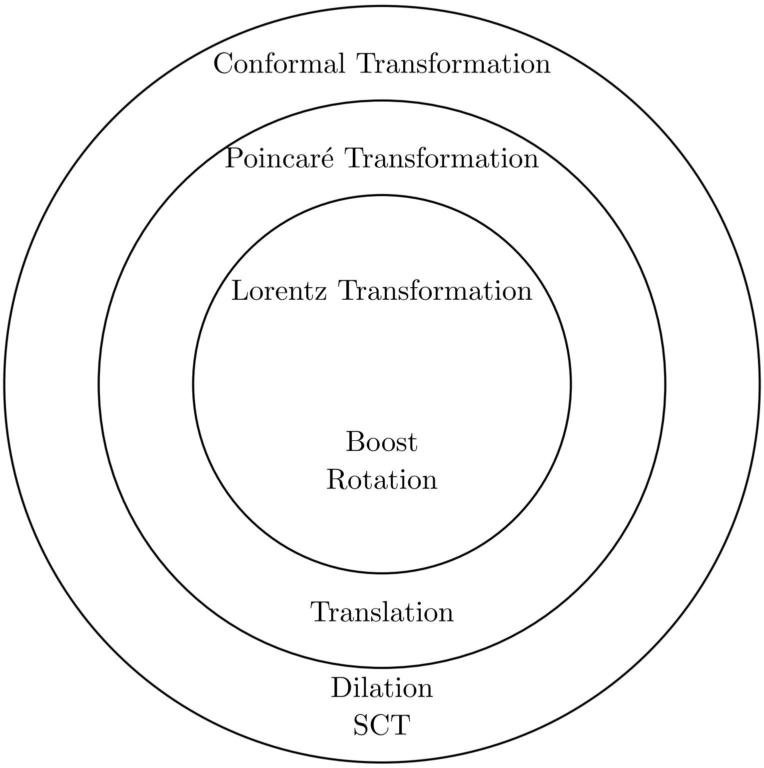

Figure 1: Subset Relationship: Conformal, Poincaré, Lorentz transformations and their symmetries. SCT stands for Special Conformal Transformation.

As mentioned before In Quantum Field Theory, fields, and particles are described by field operators that transform under the representations of the Poincare group. It turns out scale transformations that change our notion of distance are important generators of the Conformal Group.

2.2 Conformal Group

All we have said so far is transformations that transform as follows is called conformal transformation

$$\eta_{\rho\sigma}(x')\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x)\,\eta_{\mu\nu}(x) \tag{2.3}$$

But what changes satisfy this equation is unclear hence identifying them is our next interest. Let us next study infinitesimal transformation on x, $x \to x'$ up to first order

$$x'^{\rho} = x^{\rho} + \epsilon^{\rho}(x) + \mathcal{O}\left(\epsilon^{2}\right) \tag{2.4}$$

When we make this infinitesimal change to x and plug it into our equation (2.3) we get the following steps

$$\eta_{\rho\sigma}\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \eta_{\rho\sigma}\left(\delta^{\rho}_{\mu} + \frac{\partial \epsilon^{\rho}}{\partial x^{\mu}} + \mathcal{O}\left(\epsilon^{2}\right)\right)\left(\delta^{\sigma}_{\nu} + \frac{\partial \epsilon^{\sigma}}{\partial x^{\nu}} + \mathcal{O}\left(\epsilon^{2}\right)\right) \tag{2.5a}$$

$$= \eta_{\mu\nu} + \eta_{\mu\sigma}\frac{\partial \epsilon^{\sigma}}{\partial x^{\nu}} + \eta_{\rho\nu}\frac{\partial \epsilon^{\rho}}{\partial x^{\mu}} + \mathcal{O}\left(\epsilon^{2}\right) \tag{2.5b}$$

$$= \eta_{\mu\nu} + \left(\frac{\partial \epsilon_{\mu}}{\partial x^{\nu}} + \frac{\partial \epsilon_{\nu}}{\partial x^{\mu}}\right) + \mathcal{O}\left(\epsilon^{2}\right) \tag{2.5c}$$

The last step demands the second term to be equal the to metric scaled by a function

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = K(x)\eta_{\mu\nu} \tag{2.6}$$

$K(x)$ is some function of x. Multiplying by an inverse metric we have

$$\eta^{\mu\nu}\left(\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu}\right) = K(x)\eta^{\mu\nu}\eta_{\mu\nu} \tag{2.7}$$

$$2\partial^{\mu}\epsilon_{\mu} = K(x)d \tag{2.8}$$

$$K(x) = \frac{2}{d}(\partial\cdot\epsilon) \tag{2.9}$$

where d stands for dimensions of the metric. substituting equation (2.9) in (2.6) we get

$$\boxed{\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = \frac{2}{d}(\partial\cdot\epsilon)\eta_{\mu\nu}} \tag{2.10}$$

Since we are dealing with first-order infinitesimal transformation we can say the scale factor is

$$\boxed{\Lambda(x) = 1 + \frac{2}{d}(\partial\cdot\epsilon) + \mathcal{O}\left(\epsilon^{2}\right)} \tag{2.11}$$

2.3 Conformal Group at different dimensions

2.3.1 $d >= 3$

The next step would be understanding what ϵ satisfies equation (2.10). We do that by taking take a partial derivative ∂_ν and permuting indices we get

$$\partial_\rho\partial_\mu\epsilon_\nu + \partial_\rho\partial_\nu\epsilon_\mu = \frac{2}{d}\eta_{\mu\nu}\partial_\rho(\partial\cdot\epsilon), \tag{2.12a}$$

$$\partial_\nu\partial_\rho\epsilon_\mu + \partial_\mu\partial_\rho\epsilon_\nu = \frac{2}{d}\eta_{\rho\mu}\partial_\nu(\partial\cdot\epsilon), \tag{2.12b}$$

$$\partial_\mu\partial_\nu\epsilon_\rho + \partial_\nu\partial_\mu\epsilon_\rho = \frac{2}{d}\eta_{\nu\rho}\partial_\mu(\partial\cdot\epsilon). \tag{2.12c}$$

Construct the linear combination we get

$$\partial_\mu(\partial_\nu\varepsilon_\rho + \partial_\rho\varepsilon_\nu - f\eta_{\nu\rho}) + \partial_\nu(\partial_\mu\varepsilon_\rho + \partial_\rho\varepsilon_\mu - f\eta_{\mu\rho}) - \partial_\rho(\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu - f\eta_{\mu\nu}) = 0 \tag{2.13}$$

Here $f = \frac{2}{d}(\partial\cdot\epsilon)$ .which gives us

$$2\partial_\mu\partial_\nu\varepsilon_\rho = \eta_{\nu\rho}\partial_\mu f + \eta_{\mu\rho}\partial_\nu f - \eta_{\mu\nu}\partial_\rho f. \tag{2.14}$$

Taking the trace gives us

$$2\partial^2\epsilon_\rho = (2-d)\partial_\mu f, \tag{2.15}$$

This is telling us there is something special at $d = 2$ which we will come back to later

$$\partial^2(\partial_\nu\varepsilon_\mu + \partial_\mu\varepsilon_\nu) = (2-d)\partial_\mu\partial_\nu f \tag{2.16}$$

Comparing equation (2.16) with (2.10) gives us

$$(2-d)\partial_\mu\partial_\nu f = \partial^2 f\eta_{\mu\nu} \tag{2.17}$$

Upon contraction with $\eta^{\mu\nu}$ in other words taking a trace we get

$$(d-1)\partial^2 f = 0 \tag{2.18}$$

Together with (2.17) this means $\partial_\mu\partial_\nu f = 0$ and we recall $f = \frac{2}{d}(\partial\cdot\epsilon)$. If we differentiate equation (2.14) once more and use the relation $\partial_\mu\partial_\nu f = 0$ which we just found we realize that ϵ can at most be quadratic since its third order derivative is zero. Hence the general form of ϵ that obeys conformal transformations looks like

$$\boxed{\varepsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho} \tag{2.19}$$

When we substitute (2.19) in (2.10) we see 4 key observations [11]

- a_μ is unconstrained by (2.10) since it vanished with differential which means all possible translations are allowed.
- Lorentz transformation can be obtained from $b_{\mu\nu}$. $b_{\mu\nu} = \lambda\eta_{\mu\nu} + \omega_{\mu\nu}$ and we know when $\omega_{\mu\nu} = -\omega_{\nu\mu}$ it generates Lorentz group.
- The λ generates infinitesimal scale transformation.
- $c_{\mu\nu\rho} = \eta_{\mu\rho}b_\nu + \eta_{\mu\nu}b_\rho - \eta_{\nu\rho}b_\mu$ for a constant vector b_μ. These transformations are called special conformal transformations.

The conformal group has these 4 symmetries [11]

$$\begin{aligned}
&\text{Translations: } x'^\mu = x^\mu + a^\mu \\
&\text{Lorentz: } x'^\mu = \Lambda^\mu{}_\nu x^\nu \\
&\text{Dilatations (scale transformations): } x'^\mu = \lambda x^\mu \\
&\text{Special Conformal transformation: } x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2x^2}
\end{aligned}$$

2.3.2 $d = 1$

From equation (2.18) we know $(d-1)\partial^2 f = 0$ so at $d = 1$ there is no constraint on f. Therefore all smooth transformations are part of the conformal group in one dimension. Moreover, there is no idea of angles in 1D so at $d = 1$ all transformations are trivially conformal.

2.3.3 $d = 2$

We ask the same question but for $d = 2$ now. Equation (2.10) creates the constraints for ϵ. As a reminder here is the equation

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \frac{2}{d}(\partial\cdot\epsilon)\eta_{\mu\nu} \tag{2.20}$$

For two dimensions in Euclidean metric, we get the condition

$$\partial_0\epsilon_0 = +\partial_1\epsilon_1, \quad \partial_1\epsilon_0 = -\partial_0\epsilon_1 \tag{2.21}$$

This is precisely the Cauchy–Riemann equations from complex analysis. This motivates the use of complex coordinates with following translation rules

$$\begin{aligned}
z &= x^0 + ix^1, \quad \epsilon = \epsilon^0 + i\epsilon^1, \quad \partial_z = \tfrac{1}{2}\left(\partial_0 - i\partial_1\right), \\
\bar{z} &= x^0 - ix^1, \quad \bar{\epsilon} = \epsilon^0 - i\epsilon^1, \quad \partial_{\bar{z}} = \tfrac{1}{2}\left(\partial_0 + i\partial_1\right).
\end{aligned} \tag{2.22}$$

As $\epsilon(z)$ is holomorphic (a complex-valued function that is complex-differentiable at every point in its domain) so will be $f(z) = z + \epsilon(z)$ which gives rise to a two-dimensional conformal transformation $z \mapsto f(z)$. This provides us insight into how the metric transforms in 2D Conformal Groups

$$ds^2 = dzd\bar{z} \to \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}}dzd\bar{z} \tag{2.23}$$

From which we infer the scale factor as $|\frac{\partial f}{\partial z}|^2$. For the rest of the material, we will stick to the 2-D Conformal Group.

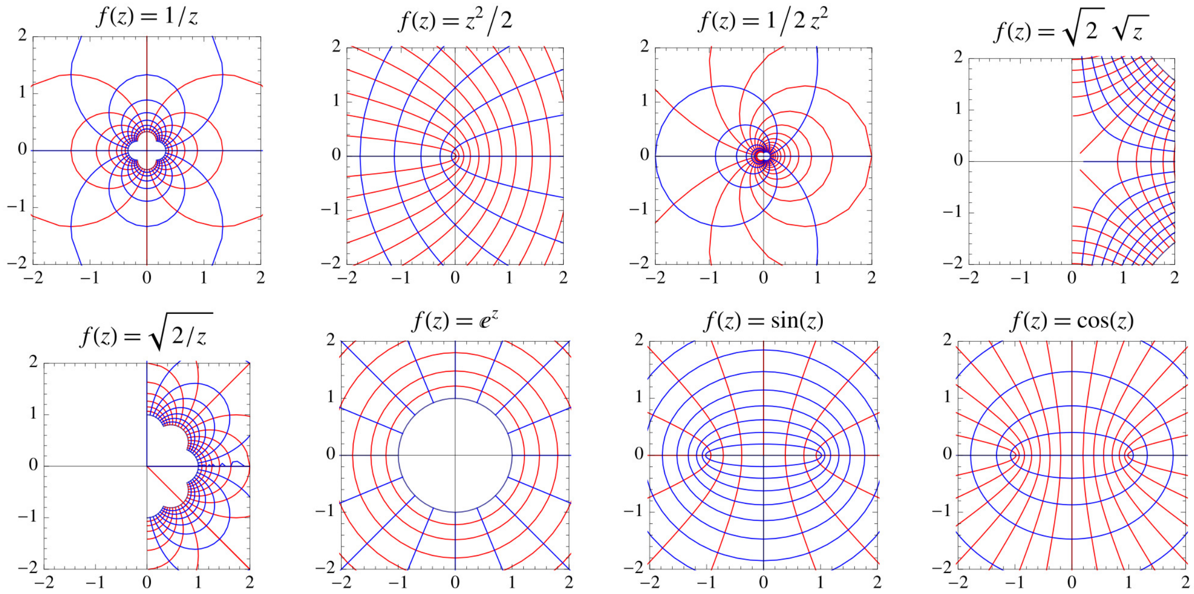

Figure 2: Examples of Conformal Transformations.Image Source https://mathworld.wolfram.com/ConformalMapping.html

2.4 Primary, Quasi-Primary and Scaling Fields

The success of Quantum Field theory is the ability to make this very bold statement

> *"Every particle and every wave in the universe is simply an excitation of a quantum field that is defined over all space and time." [15]*

The idea eventually led to very precise predictions in QED.

In 2D CFTs Primary fields correspond to local operators in the theory (Primary operators are a subset of local operators with special transformation properties under conformal transformations). This means that each primary field represents a distinct physical quantity that can be measured at each point in space-time. Correlation functions in 2D CFTs can often be computed exactly, making it an exciting topic to explore.

So what are primary fields? Under conformal transformation, if the field transforms this way it is called a primary field [11]

$$\boxed{\phi(z,\bar{z}) \mapsto \phi'(z,\bar{z}) = \left(\frac{\partial f}{\partial z}\right)^{h}\left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}}\phi(f(z),\bar{f}(\bar{z}))} \tag{2.24}$$

This is a form very similar to the one derived for the 2D conformal group in Equation (2.23). All ϕs that satisfy the above relation are **primary fields**.

Depending on the nature of the function f we can further classify fields under conformal transformations. If the field relations are true for any function f then the fields are primary.

If the relation (2.24) is only true for a smaller set of functions f namely where $f \in \mathrm{SL}(2, \mathbb{C})/\mathbb{Z}_2$ it's called a **quasi-primary field**. Quasi-primary field transforms covariantly under translations, dilatations, and rotations, but not necessarily under special conformal transformations. All primary fields are quasi-primary, but not all quasi-primary fields are primary.

Scaling fields are those fields that transform in a certain way under scaling transformations(dilatations).

Specifically, a scaling field $\Phi(x)$ of scaling dimension Δ transforms as:

$$\Phi'(\lambda x) = \lambda^{-\Delta}\Phi(x) \tag{2.25}$$

under the scaling transformation $x \rightarrow \lambda x$. Here, λ is a real number.

Every primary field and quasi-primary field is also a scaling field, with the scaling dimension given by the conformal dimension of the field

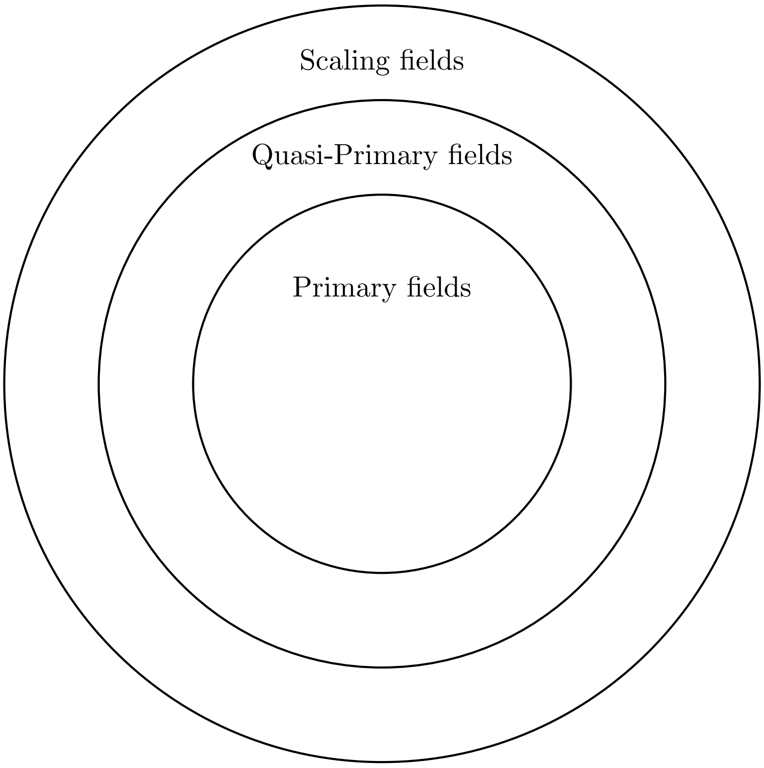

Figure 3: Subset Relationship: Scalaing, Quasi-Primary, and Primary fields

2.5 Energy–Momentum Tensor

Symmetries are interesting what's more interesting is the Energy–Momentum Tensor which helps us move from symmetries to conserved quanties.

Conformal symmetry $x^{\mu} \to x^{\mu} + \epsilon^{\mu}(x)$ should give us a conserved current (Noether's theorem) $\partial^{\mu} j_{\mu} = 0$. We can write the current as [11]

$$j_{\mu} = T_{\mu\nu}\epsilon^{\nu} \tag{2.26}$$

$T_{\mu\nu}$ is a symmetric energy–momentum tensor. Differentiating and setting (2.25) to zero (conservation) gives us the following steps

$$\begin{aligned} 0 = \partial^{\mu} j_{\mu} &= \left(\partial^{\mu} T_{\mu\nu}\right) \epsilon^{v} + T_{\mu\nu}\left(\partial^{\mu}\epsilon^{\nu}\right) \\ &= 0 + \frac{1}{2} T_{\mu\nu}\left(\partial^{\mu}\epsilon^{v} + \partial^{\nu}\epsilon^{\mu}\right) = \frac{1}{2} T_{\mu v}\eta^{\mu v}(\partial\cdot\epsilon)\frac{2}{d} = \frac{1}{d} T^{\mu}_{\mu}(\partial\cdot\epsilon) \end{aligned} \tag{2.27}$$

We have used the result from (2.10) and $T_{\mu\nu}$ is symmetric to arrive at the final answer.

$$\frac{1}{d} T^{\mu}_{\mu}(\partial\cdot\epsilon) = 0 \tag{2.28}$$

We have not made any assumptions for ϵ other than that is required for conformal transformations. Therefore we come and an important conclusion about energy-momentum tensors in conformal field theory

$$\boxed{\mathbf{T}^{\mu}_{\mu} = 0 \quad \text{Energy-momentum tensor is traceless in conformal field theory.}}$$

These have important implications for 2D on energy-momentum tensor components. To show that we first make coordinate transformation from real to complex we find [11] (for T_{zz} we used that $\eta_{\mu\nu} = \operatorname{diag}(+1,+1)$) [11]

$$T_{zz} = \frac{1}{4}(T_{00} - 2iT_{10} - T_{11}) \tag{2.29}$$

$$\bar{T}_{\bar{z}\bar{z}} = \frac{1}{4}\left(T_{00} + 2iT_{10} - T_{11}\right), \tag{2.30}$$

$$T_{zz} = \bar{T}_{\bar{z}\bar{z}} = \frac{1}{4}\left(T_{00} + T_{11}\right) = \frac{1}{4} T_{\mu\mu} = 0 \tag{2.31}$$

$$T_{zz} = \frac{1}{2}\left(T_{00} - iT_{10}\right) \tag{2.32}$$

$$\bar{T}_{\bar{z}\bar{z}} = \frac{1}{2}\left(T_{00} + iT_{10}\right). \tag{2.33}$$

Since this has to be translational invariant we get

$$0T_{00} + 1T_{10} = 0, \quad 0T_{01} + 1T_{11} = 0, \quad (2.22) \tag{2.34}$$

Which gives us

$$\partial_{z} T_{zz} = \frac{1}{4}\left(\partial_{0} + i\partial_{1}\right)\left(T_{00} - iT_{10}\right) = \frac{1}{4}\left(\partial_{0} T_{00} + \partial_{1} T_{10} + i\partial_{1} T_{00} - i\partial_{0} T_{10}\right) = 0 \tag{2.35}$$

$$\boxed{T_{zz}(z,\bar{z}) =: T(z), T_{\bar{z}\bar{z}}(z,\bar{z}) =: \bar{T}(\bar{z})} \tag{2.36}$$

The above two fields are the only two non-vanishing components for 2D energy-momentum tensor components as a consequence of energy-momentum tensor being traceless.

2.6 The Witt Algebra

In the context of infinitesimal conformal transformations a general representation of an infinitesimal conformal transformation as [11]

$$z' = z + \epsilon(z) = z + \sum_{n\in\mathbb{Z}} \epsilon_n(-z^{n+1}) \tag{2.37}$$

$$\bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z}) = \bar{z} + \sum_{n\in\mathbb{Z}} \bar{\epsilon}_n(-\bar{z}^{n+1}) \tag{2.38}$$

The generators corresponding to a transformation for a specific n are defined as [11]

$$l_n = -z^{n+1}\partial_z \quad \text{and} \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \tag{2.39}$$

It is important to note that since $n \,\epsilon\, Z$, an **infinite** number of independent infinitesimal conformal transformations are possible. This is a special result for two dimensions.

To find the corresponding algebra, we compute the commutators of the generators, yielding [11]

As a next step, let us compute the commutators of the generators (2.11) in order to determine the corresponding algebra. We calculate

$$[l_m, l_n] = z^{m+1}\partial_z(z^{n+1}\partial_z) - z^{n+1}\partial_z(z^{m+1}\partial_z) \tag{2.40}$$

$$= (n+1)z^{m+n+1}\partial_z - (m+1)z^{m+n+1}\partial_z \tag{2.41}$$

$$= -(m-n)z^{m+n+1}\partial_z = (m-n)l_{m+n}, \tag{2.42}$$

$$[l_m, l_n] = (m-n)l_{m+n} \tag{2.43}$$

$$[\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n} \tag{2.44}$$

$$[l_m, \bar{l}_n] = 0 \tag{2.45}$$

This analysis reveals that the algebra of infinitesimal conformal transformations in a Euclidean two-dimensional space is infinite-dimensional.

2.7 Virasoro Algebra and Descendant Fields

In the study of 2D CFTs, the Virasoro algebra plays a pivotal role as an extension of the Witt algebra, introducing a central charge that becomes crucial in the quantization of the theory. To get the Virasoro algebra from the Witt algebra, we introduce a central extension. A central extension is an extension of an algebra by adding a term that commutes with all other elements in the algebra (a central element).

The Virasoro generators L_n can be expressed in terms of the energy-momentum tensor, providing a bridge between the symmetries of the theory and the dynamics of the fields.

The Virasoro algebra is given by the commutation relations [11]

$$\boxed{[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}} \tag{2.46}$$

where c is the central charge, a crucial parameter in the theory, and L_m are the Virasoro generators.

Descendant fields are constructed by acting with the Virasoro generators on the primary fields. For instance, a level one descendant is created as $L_{-1}\phi(z)$, and a level two descendant can be constructed as $L_{-2}\phi(z)$ or $L_{-1}L_{-1}\phi(z)$. These descendant fields inherit certain properties from the primary fields, and their behavior under conformal transformations can be determined using the Virasoro algebra.

The inclusion of descendant fields in the theory enriches the space of states and allows for a more detailed exploration of the conformal structure of the theory. Moreover, the Virasoro algebra and descendant fields play a crucial role in the computation of correlation functions, particularly in determining the operator product expansion (OPE) of primary fields, which can be expressed in terms of primary and descendant fields.

2.8 Ground State in Conformal Field Theory

In Conformal Field Theory, the ground state, often denoted as $|0\rangle$ or $|\Omega\rangle$, is defined as the state that is annihilated by the positive modes of the Virasoro generators and the mode L_0:

$$L_n|0\rangle = 0, \text{ for } n \geq 0 \tag{2.47}$$

The ground state in a CFT possesses several important properties:

- **Lowest Energy State**: It represents the state of the system with the lowest energy, serving as the vacuum state in the theory, around which particle excitations and other states are defined.
- **Invariant under Conformal Transformations**: The ground state retains its form under the action of the conformal group, a property crucial in maintaining the conformal symmetry of the theory.

- **Building Block for Other States**: The ground state serves as a foundational element for constructing other states in the theory, with other states being obtained by acting on the ground state with the negative modes of the Virasoro generators (creation operators).
- **Unique Scaling Dimension**: The ground state has a unique scaling dimension, which is zero, reflecting its role as the lowest energy state in the theory.

2.8.1 Role in Correlation Functions

In the computation of correlation functions, the ground state often appears as the state upon which the field operators act. The correlation functions are computed as expectation values with respect to this ground state, offering insights into the behavior of fields and their interactions in the theory.

3 Quantum Information

Let's for a moment diverge from CFTs and understand some fundamentals of quantum information.

In his book "Decoding Reality" Oxford professor Vlatko Vedral mentions an important fact about information. "Information is Physical" [8] This fact that all information is physical connects the field of Information theory with the study of physical systems a.k.a physics. Hence understanding the fundamental physics is also understanding the fundamentals of information theory.

3.1 Information

In classical information theory, information is typically represented in bits, which can take the value of either 0 or 1. However, in quantum information theory, the basic unit of information is called a quantum bit or qubit.

A simple question to ask in classical information theory would be to ask "How much information is there in a fair coin?"

The answer is 1 bit it's either Heads or Tails (0 or 1). Then one can ask "How much information is there in a given quantum state?". We will try to answer this soon after reviewing some concepts.

3.2 Density Matrices

An alternate formulation for quantum mechanics other than the state vector formalism is using a tool known as the density operator or density matrix. Here we are replacing the idea of the state usually denoted $|\psi\rangle$ with an operator

$$\boxed{\rho = |\psi\rangle\langle\psi|} \tag{3.1}$$

The density matrix formalism can describe mixed states, which are statistical mixtures of different quantum states. This is particularly useful in quantum statistical mechanics and in situations where we have incomplete information about the system.

The density matrix formalism also allows for a straightforward description of subsystems through the partial trace operation. This makes it easier to analyze subsystems of composite systems, including those that are entangled with other subsystems.

3.2.1 Mixed Bell sates

We'll consider two Bell states example to illustrate the density matrix formalism, $|\Psi^+\rangle$ and $|\Psi^-\rangle$, defined as [10]

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

Next, we construct the density matrices for each of these states:

$$\rho_{\Psi^+} = |\Psi^+\rangle\langle\Psi^+| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\rho_{\Psi^-} = |\Psi^-\rangle\langle\Psi^-| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Now, let's assume we have an ensemble where the system is in state $|\Psi^+\rangle$ with probability p and in state $|\Psi^-\rangle$ with probability $1-p$. We construct the mixed density matrix as a weighted sum of the density matrices for $|\Psi^+\rangle$ and $|\Psi^-\rangle$:

$$\rho = p\rho_{\Psi^+} + (1-p)\rho_{\Psi^-} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 2p-1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2p-1 & 0 & 0 & 1 \end{pmatrix}$$

This mixed density matrix ρ describes an ensemble of the two Bell states. The diagonal elements represent the probabilities of finding the system in the respective basis states ($|00\rangle$, ($|01\rangle$, ($|10\rangle$ and $|11\rangle$), and the off-diagonal elements represent the coherence between these states, which depends on the probabilities p and $1-p$.

3.3 Entropy

Entropy is a fundamental concept in both classical and quantum physics. It represents the degree of disorder or randomness in a system.

3.3.1 Shannon Entropy

Shannon entropy is a concept from information theory that quantifies the amount of uncertainty or randomness in a set of data. For a discrete random variable X with possible outcomes $x_1, x_2, ..., x_n$ and a probability mass function $P(x)$, the Shannon entropy H(X) is given by [10]

$$H(X) = -\sum P(x_i) \log_2 P(x_i) \tag{3.2}$$

For a fair coin, we have $H(X) = -[P(H) \log_2 P(H) + P(T) \log_2 P(T)]$ since both the probabilities are 0.5 This gives us 1 which means 1 Bit is of information is needed to clear the uncertainty.

3.3.2 Von Neumann entropy

The Von Neumann entropy is a measure of uncertainty or disorder for a quantum system, analogously to how the Shannon entropy measures these quantities for a classical system. For a quantum system described by a density matrix ρ, the Von Neumann entropy (S) is defined as [10]

$$S = -\mathrm{Tr}(\rho \ln(\rho)) \tag{3.3}$$

where Tr represents the trace operation. This formula captures the uncertainty or 'mixedness' in the state of a quantum system.

3.3.3 Rényi entropy

The Rényi entropy is a generalization of the Shannon entropy and the von Neumann entropy. It is parametrized by a number q and can be used to quantify the diversity, uncertainty, or randomness of a system.

Rényi entropy of a discrete probability distribution [10]

$$H_q = \frac{1}{1-q} \log \left(\sum p_i^q \right) \tag{3.4}$$

For the Rényi entropy of a quantum state:

$$S_q = \frac{1}{1-q} \log \left(\mathrm{Tr} \left(\rho^q \right) \right) \tag{3.5}$$

In the limit q → 1 we get Von Neumann entropy and Shannon Entropy.

3.4 Entanglement

Entanglement is a unique phenomenon where two or more particles become interconnected such that the state of one particle is instantaneously correlated with the state of the other, regardless of the distance between them. This property has significant implications for quantum computing and quantum information theory.

One way to measure the entanglement of a quantum system is by using the concept of entanglement entropy.

3.5 Entanglement Entropy

Entanglement entropy is a measure of the amount of entanglement between two parts of a quantum system. If we have a quantum system divided into two parts A and B, the entanglement entropy of part A is defined as the von Neumann entropy of the reduced density matrix of A.

$$S(A) = -\mathrm{Tr}(\rho_A \log(\rho_A)) \tag{3.6}$$

And for defining the reduced density matrix ρ_A:

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) \tag{3.7}$$

The entanglement entropy gives a measure of the "amount" of entanglement between subsystem A and the rest of the system. If the entanglement entropy is zero, then the subsystem A is not entangled with B, i.e., the total system can be described by a product state. If the entanglement entropy is nonzero, it indicates that the state of the system is entangled.

If systems are not entangled (can be written as the product of states) we can show Von Neumann entropy is zero.

Let's consider a product state ρ_{AB} which can be written as a tensor product of two separate states ρ_A and ρ_B, i.e.,

$$\rho_{AB} = \rho_A \otimes \rho_B \tag{3.8}$$

Now, we want to find $S(A)$, which is given by

$$S(A) = -\mathrm{Tr}(\rho_A \log(\rho_A)) \tag{3.9}$$

But first, we need to find ρ_A, which is given by tracing out the B part from ρ_{AB}, i.e.,

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) \tag{3.10}$$

For product states, this tracing operation simply gives us back ρ_A, because there are no correlations between A and B. So,

$$\rho_A = \mathrm{Tr}_B(\rho_A \otimes \rho_B) = \rho_A \tag{3.11}$$

Now, we can find $S(A)$:

$$S(A) = -\mathrm{Tr}(\rho_A \log(\rho_A)) \tag{3.12}$$

Since ρ_A is a pure state (as it is part of a product state), its logarithm will be zero, and hence $S(A)$ will be zero:

$$S(A) = -\mathrm{Tr}(\rho_A \log(\rho_A)) = -\mathrm{Tr}(\rho_A \cdot 0) = 0 \tag{3.13}$$

So, we have shown that $S(A) = 0$ for product states.

If $S(A)$ and $S(B)$ are von Neumann entropy of two subsystem Relationship between $S(A)$ and $S(B)$ is as follows

3.5.1 For Product States

For product states where

$$\rho_{AB} = \rho_A \otimes \rho_B \tag{3.14}$$

the entropies $S(A)$ and $S(B)$ are independent of each other, and are given by

$$S(A) = -\mathrm{Tr}(\rho_A \log(\rho_A)), \quad S(B) = -\mathrm{Tr}(\rho_B \log(\rho_B)) \tag{3.15}$$

3.5.2 For Entangled States

For entangled states, the subsystems A and B are correlated, and the entropies $S(A)$ and $S(B)$ can be related through the properties of the joint state ρ_{AB}. In particular, for pure entangled states where $\rho_{AB} = |\Psi\rangle\langle\Psi|$ for some state $|\Psi\rangle$, the entropies of the subsystems are equal, i.e.,

$$S(A) = S(B) \tag{3.16}$$

This is a manifestation of the entanglement between A and B.

3.5.3 Conditional Entropy

The relationship between $S(A)$ and $S(B)$ can also be explored through the concept of conditional entropy, which is defined as [10]

$$S(A|B) = S(AB) - S(B) \tag{3.17}$$

where $S(AB)$ is the entropy of the joint system. The conditional entropy $S(A|B)$ represents the uncertainty about A given B.

3.5.4 Mutual Information

Another quantity that relates $S(A)$ and $S(B)$ is the mutual information, which is a measure of the total correlation between A and B, and is given by [10]

$$I(A;B) = S(A) + S(B) - S(AB) \tag{3.18}$$

3.6 Entanglement Hamiltonian

When we divide a quantum system into two parts A and B, you can describe the state of part A by a reduced density matrix ρ_A obtained by taking the partial trace over the degrees of freedom in part B from the total system's density matrix. The entanglement Hamiltonian $\mathbf{H_E}$ is defined through this reduced density matrix ρ_A up to a constant in the following way [5]

$$\rho_A = \frac{e^{-H_E}}{Z} \tag{3.19}$$

This equation is not valid when the eigenvalues are zero. Z is the partition function (which ensures that $Tr(\rho_A) = 1$) defined as

$$Z = \mathrm{Tr}(e^{-H_E}) \tag{3.20}$$

3.7 Pure Vs Mixed states

Pure Vs Mixed states are often confusing and makes understanding concepts like Von Neumann's entropy difficult so we spend this section clarifying the difference.

A quantum state of a single qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where $|\alpha|^2 = |\beta|^2 = 1/2$ is still a pure state. This is because it can be described by a single wave function, even though the probabilities of finding the system in the $|0\rangle$ or $|1\rangle$ state upon measurement are equal (1/2 each). This pure state is a superposition of $|0\rangle$ and $|1\rangle$ states.

On the other hand, a mixed state is a statistical mixture of multiple distinct states. An example of a mixed state is a system that is in state $|0\rangle$ with probability 1/2 and in state $|1\rangle$ with probability 1/2. This doesn't mean it's a superposition of these states (which would be a pure state), but rather that we don't know which one of those states the system is in, and there's equal probability for either.

The von Neumann entropy measures the "mixedness" or uncertainty of a quantum state. For a pure state, even one where $|\alpha|^2 = |\beta|^2 = 1/2$, the von Neumann entropy is zero. This is because there's no uncertainty about the state itself - it's definitely in the state $\alpha|0\rangle + \beta|1\rangle$. However, for a mixed state where the system is in $|0\rangle$ with probability 1/2 and in $|1\rangle$ with probability 1/2, the von Neumann entropy is 1, reflecting our complete lack of knowledge about which state the system is actually in.

Let's see the calculation

Consider a pure state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Its density matrix is:

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\rho = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

To calculate the eigenvalues, we need to solve the characteristic equation, which is given by $det(\rho - \lambda I) = 0$, where I is the identity matrix.

So, we subtract λ times the identity matrix from ρ:

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 - \lambda & 1/2 \\ 1/2 & 1/2 - \lambda \end{bmatrix}$$

Now, we calculate the determinant and set it equal to 0:

$$(1/2 - \lambda)^2 - (1/2)(1/2) = 0$$

Solving this quadratic equation gives the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 1$.

So the eigenvalues of this matrix are 0 and 1. The von Neumann entropy is calculated as follows:

$$S(\rho) = -\sum_i \lambda_i \log_2(\lambda_i) = -(0 \cdot \log_2(0) + 1 \cdot \log_2(1))$$

In mathematics, we define $0 \cdot \log_2(0)$ to be 0, as the limit of $x \log_2(x)$ as x approaches 0 is 0. So, the von Neumann entropy is:

$$S(\rho) = -(0 + 0) = 0$$

So, for this **pure state**, even though it's a superposition of two basis states with equal probability, the von Neumann entropy is 0, **indicating it's a pure state, not mixed**.

Next, consider a **mixed state** that is in the state $|0\rangle$ with probability 1/2 and in the state $|1\rangle$ with probability 1/2. Its density matrix is:

$$\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rho = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Again, we subtract λ times the identity matrix from ρ:

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 - \lambda & 0 \\ 0 & 1/2 - \lambda \end{bmatrix}$$

The determinant of this matrix is $(1/2 - \lambda)^2 = 0$, which gives the eigenvalues $\lambda_1 = 1/2$ and $\lambda_2 = 1/2$.

The von Neumann entropy is calculated as follows:

$$S(\rho) = -\sum_i \lambda_i \log_2(\lambda_i) = -(\frac{1}{2}\log_2(\frac{1}{2}) + \frac{1}{2}\log_2(\frac{1}{2}))$$

So, the von Neumann entropy is:

$$S(\rho) = -(-\frac{1}{2} - \frac{1}{2}) = 1$$

For this mixed state, the von Neumann entropy is 1, **indicating complete uncertainty about which state the system is in**.

3.8 Entropy and Entanglement Hamiltonian of simple systems

The goal of these examples is to make the concepts of **Entropy** and **Entanglement Hamiltonian** clear and we do that now by considering a simple two-entangled qubit system.

3.8.1 Pure state

We choose Bell state (also called EPR pair) a maximally entangled two-qubit system.

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

The density matrix as we know is defined as

$$\rho = |\Psi\rangle\langle\Psi| = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$$

In matrix form, this is:

$$\rho = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

The eigenvalues of this matrix are 0, 0, 0, and 1. To calculate the von Neumann entropy, we use the formula:

$$S(\rho) = -\sum_i \lambda_i \log_2(\lambda_i)$$

$$S(\rho) = -(0 * \log_2(0) + 0 * \log_2(0) + 0 * \log_2(0) + 1 * \log_2(1)) = 0$$

So, for this entangled state, the von Neumann entropy is 0, indicating it's a **pure state**.

Let's see what is the **Entanglement entropy** of this system. As mentioned before The entanglement entropy of a density matrix ρ is defined as the von Neumann entropy of the reduced density matrix of a subsystem.

We treat each qubit as system A and B. Then we trace out the second qubit B, we obtain the reduced density matrix for the first qubit A

$$\rho_A = \mathrm{Tr}_B[\rho]$$

$$\rho_A = \frac{1}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \frac{1}{2}I$$

$$S(\rho_A) = -\sum_{i=1}^{2}\lambda_i \log_2 \lambda_i = -2\cdot\frac{1}{2}\log_2\frac{1}{2} = 1$$

So, for this state, the entanglement entropy is 1. This is the maximum possible value for a two-level system, which means that the two qubits are **maximally entangled**.

The **Entanglement Hamiltonian** H_A is defined such that the reduced density matrix ρ_A for a subsystem A is given by $\rho_A = e^{-H_A}$, where H_A is the entanglement Hamiltonian.

For our two-qubit system, we obtained the reduced density matrix ρ_A as:

$$\rho_A = \frac{1}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \frac{1}{2}I$$

To find the entanglement Hamiltonian, we need to solve the equation $\rho_A = e^{-H_A}$.

Taking the natural logarithm of both sides, we have:

$$H_A = -\ln(\rho_A)$$

Since ρ_A is a diagonal matrix, its natural logarithm is also diagonal, with the natural logarithm of each eigenvalue on the diagonal.

As $\ln(\frac{1}{2}) = -\ln(2)$, we find that the entanglement Hamiltonian H_A in this case is:

$$H_A = \ln(2)\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \ln(2)I$$

The result we obtained reflects the **maximal uncertainty of the reduced state**, in accordance with the maximally mixed nature of the reduced density matrix.

3.8.2 Mixed state

We follow the same procedure as a mixed state

Density matrix for a mixed state of two qubits:

$$\rho = \frac{1}{4}\left(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|\right)$$

Matrix representation:

$$\rho = \frac{1}{4}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

The reduced-density matrix

$$\rho_A = \mathrm{Tr}_B[\rho]$$

For our state, this gives:

$$\rho_A = \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Then we calculate Entanglement Hamiltonian:

$$H_A = -\ln(\rho_A)$$

This gives:

$$H_A = \ln(2)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \ln(2)I$$

In this case, the entanglement Hamiltonian is the same for both the mixed and pure state examples because the reduced density matrix ρ_A is the same in both cases. However, this doesn't mean that the two original states were the same – one was a maximally entangled pure state, and the other was a maximally mixed state. The entanglement Hamiltonian only gives us information about the state of a subsystem (in this case, one of the qubits) after tracing out the other subsystem(s). The original global states (before taking the partial trace) could be very different.

3.8.3 Pure state but not maximally entangled state

We try to show an example where entanglement Hamiltonian is not proportional to the identity matrix.

Consider a two-qubit system in the following pure entangled state:

$$|\psi\rangle = \frac{\sqrt{3}}{2}|00\rangle + \frac{1}{2}|11\rangle$$

The density matrix for this state is:

$$\rho = |\psi\rangle\langle\psi| = \frac{3}{4}|00\rangle\langle 00| + \frac{1}{4}|11\rangle\langle 11| + \frac{\sqrt{3}}{4}(|00\rangle\langle 11| + |11\rangle\langle 00|)$$

This can be represented as:

$$\rho = \begin{bmatrix} \frac{3}{4} & 0 & 0 & \frac{\sqrt{3}}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\sqrt{3}}{4} & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

We can calculate the reduced density matrix by tracing out subsystem B:

$$\rho_A = \mathrm{Tr}_B[\rho]$$

This gives us:

$$\rho_A = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

Finally, the entanglement Hamiltonian is given by:

$$H_A = -\log(\rho_A)$$

To compute the logarithm of a matrix, we can diagonalize the matrix, apply the logarithm to the eigenvalues, and then transform it back. In this case, the matrix ρ_A is already diagonal, so this is straightforward:

$$H_A = \begin{bmatrix} -\ln\left(\frac{3}{4}\right) & 0 \\ 0 & -\ln\left(\frac{1}{4}\right) \end{bmatrix}$$

This results in:

$$H_A = \begin{bmatrix} 0.28768 & 0 \\ 0 & 1.38629 \end{bmatrix}$$

This entanglement Hamiltonian is not proportional to the identity matrix, reflecting the fact that our original state $|\psi\rangle$ was **not a maximally entangled state**. The reason we discuss the proportionality to the identity matrix in detail is because that is one of the main proposals of the paper we review.

4 Conformal Field Theory and Quantum Information

After a brief review of CFTs and Quantum Information, we discuss some of the concepts where we bring these two concepts.

Entanglement Entropy for a 1+1D ground state CFT on an interval $[x_1, x_2]$ is given by [1]

$$S[x_1, x_2] = \frac{c}{3} \frac{\log(x_2 - x_1)}{\epsilon} \tag{4.1}$$

where c is the central charge, UV cutoff ϵ.

Entanglement Hamiltonion for a 1+1D ground state CFT on an interval $[x_1, x_2]$ is given by [4]

$$K[x_1, x_2] = 2\pi \int_{x_1}^{x_2} dx \frac{(x - x_1)(x_2 - x)}{x_2 - x_1} T_{00}(x) + \text{ const.} \tag{4.2}$$

T is the stress-energy tensor, and const. is a number that depends on the UV cutoff ϵ.

4.1 Conformal Field Theory Ground States as Critical Points of an Entropy Function

In this section, we use all the discussions from above and review the recent paper [9] titled **"Conformal Field Theory Ground States as Critical Points of an Entropy Function"** The main result of the paper is as follows. Let A, B, and C be three consecutive intervals;

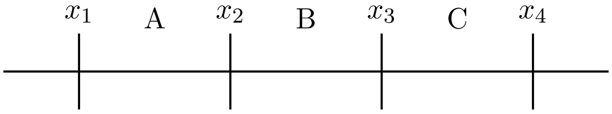

Figure 4: Three consecutive intervals on an infinite system with the corresponding cross ratio η [9] (shown below). Here, $x_{ij} := x_j - x_i$ denotes the distance between x_i and x_j .

$$\eta = \frac{(x_2 - x_1)(x_4 - x_3)}{(x_3 - x_1)(x_4 - x_2)} \tag{4.3}$$

- As mentioned above the Entanglement Hamiltonian for 1+1D CFT ground state on an interval $[x_1, x_2]$ is known to be [4]

$$K[x_1, x_2] = 2\pi \int_{x_1}^{x_2} dx \frac{(x - x_1)(x_2 - x)}{x_2 - x_1} T_{00}(x) + \text{const.} \tag{4.4}$$

  $T_{00}(x)$ is the local energy-momentum tensor.

- The paper defined K_Δ as follows [9]

$$\boxed{K_\Delta := (K_{AB} + K_{BC}) - \eta(K_A + K_C) - (1 - \eta)(K_B + K_{ABC})} \tag{4.5}$$

  Where K_A, K_{AB} are Entanglement Hamiltonian of the respective intervals.

  And Entanglement Entropy is defined as follows [9]

$$\boxed{S_\Delta(|\psi_i\rangle) := (S_{AB} + S_{BC}) - \eta(S_A + S_C) - (1 - \eta)(S_B + S_{ABC})} \tag{4.6}$$

  Where S_A, S_{AB} are Von Neumann entropy of respective intervals

4.2 Claims

There are 2 main claims this paper makes

1. $K_\Delta \propto I$ and $K_\Delta|\psi\rangle \propto |\psi\rangle$ are **satisfied by 1+1D ground state unitary CFT** $|\psi\rangle$ where K_Δ defined as

$$K_\Delta := (K_{AB} + K_{BC}) - \eta(K_A + K_C) - (1-\eta)(K_B + K_{ABC}) \tag{4.7}$$

2. $K_\Delta \propto I$ and $K_\Delta|\psi\rangle \propto |\psi\rangle$ are the conditions for $|\psi\rangle$ **to be a critical point of the entropy function** S_Δ. Where S_Δ is defined as

$$S_\Delta(|\psi\rangle) := (S_{AB} + S_{BC}) - \eta(S_A + S_C) - (1-\eta)(S_B + S_{ABC}) \tag{4.8}$$

4.3 $K_\Delta \propto I$

This condition imposes that the reduced density matrix be a maximally mixed state. We have shown examples of this before now we show that in general.

Let's take Entanglement Hamiltonian to be protional to identity I and c here is a proportionality constant.

$$K_\Delta = cI \tag{4.9}$$

And we know Entanglement Hamiltonian is defined as

$$\rho_R = e^{-K_\Delta} \tag{4.10}$$

This gives us

$$\rho_R = e^{-cI} \tag{4.11}$$

Exponential of a matrix A is defined by its Taylor series:

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \tag{4.12}$$

So for e^{-cI}, this reduces to:

$$\begin{aligned} e^{-cI} &= I - cI + \frac{(cI)^2}{2!} - \frac{(cI)^3}{3!} + \cdots \\ &= I(1 - c + \frac{c^2}{2!} - \frac{c^3}{3!} + \cdots) \\ &= Ie^{-c} \end{aligned}$$

Instead of considering e^{-c}, we replace it with a constant term α:

$$\rho_R = \alpha I \tag{4.13}$$

The equation $\rho_R = \alpha I$ represents a maximally mixed state because.

1. All the off-diagonal elements are zero hence no coherence.
2. All the diagonal elements are equal hence maximal uncertainty

The above conditions demand that the entanglement entropy of the subsystem is maximum. We first show that for a simple 2-bit example and then generalize.

4.3.1 Example for a 2-Qubit System

Let's consider a 2-qubit system. For such a system, the dimension of the reduced density matrix for one of the qubits would be $n = 2$.

We can write the reduced density matrix for one of the qubits as:

$$\rho_R = \alpha I \tag{4.14}$$

Here, I is the 2×2 identity matrix:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.15}$$

And α is a constant term that we aim to determine.

Normalising ρ_R, which means $\mathrm{Tr}(\rho_R) = 1$

$$\alpha = \frac{1}{2} \tag{4.16}$$

The reduced density matrix would be

$$\rho_R = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \tag{4.17}$$

This is the maximally mixed state for one of the qubits in a 2-qubit system.

4.3.2 Entanglement Entropy of a 2-Qubit Maximally Mixed State

The entanglement entropy is calculated using the formula

$$S = -\mathrm{Tr}(\rho_R \log_2 \rho_R) \tag{4.18}$$

For a maximally mixed state, the reduced density matrix ρ_R is (as shown before)

$$\rho_R = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.19}$$

To find $\log_2 \rho_R$, we diagonalize ρ_R and apply the logarithm to each of its eigenvalues. In this case, ρ_R is already diagonal, and its eigenvalues are $\frac{1}{2}$ and $\frac{1}{2}$. Thus,

$$\log_2 \rho_R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.20}$$

The entanglement entropy S can now be calculated as:

$$\begin{aligned} S &= -\mathrm{Tr}(\rho_R \log_2 \rho_R) \\ &= -\mathrm{Tr}\left(\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right) \\ &= -\mathrm{Tr}\left(-\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= -\left(-\frac{1}{2} - \frac{1}{2}\right) \\ &= 1 \end{aligned}$$

The entanglement entropy S for this maximally mixed state of one of the qubits in a 2-qubit system is 1.

4.3.3 Entanglement Entropy for an n-Qubit Maximally Mixed State

1. **Reduced Density Matrix**: For a maximally mixed state, the reduced density matrix ρ_R is:

$$\rho_R = \frac{1}{n} I$$

where I is the $n \times n$ identity matrix.

2. **Logarithm of Reduced Density Matrix**: The eigenvalues of ρ_R are $\frac{1}{n}$, so

$$\log_2 \rho_R = -\log_2 n\, I$$

3. **Entanglement Entropy**: The entanglement entropy S is calculated using:

$$S = -\mathrm{Tr}(\rho_R \log_2 \rho_R) = \log_2 n\, \mathrm{Tr}\left(\frac{1}{n} I\right) = \log_2 n$$

Thus, for an n-qubit maximally mixed state, the entanglement entropy is $\log_2 n$.

Which is the maximum entropy for n-qubit system. Therefore the condition $K_\Delta \propto I$ imposes maximum Entanglement entropy from the reduced density matrix.

4.4 $K_\Delta|\psi\rangle \propto |\psi\rangle$

This condition states that $|\psi\rangle$ is an eigenvector of the K_Δ operator or acting K_Δ on $|\psi\rangle$ state only scales the state by a certain factor.

4.5 Example: Single interval system

Let's take a simple single-interval system to follow the calculations and claims made in the paper.

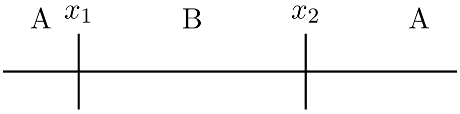

Figure 5: A single interval (B) on an infinite system. The regions outside the interval are labeled as A. Here, $x_{12} := x_2 - x_1$ denotes the distance between x_1 and x_2.

Let's see what simplifications occur by considering a single interval. Cross-ratio as shown earlier for a 3-interval system is defined as

$$\eta = \frac{(x_2 - x_1)(x_4 - x_3)}{(x_3 - x_1)(x_4 - x_2)} \tag{4.21}$$

One can think of taking x_1 and x_2 as the same points and x_3 and x_4 as the same points as well. Which reduces the 3-interval system to one interval. **In this case, the cross-ratio is zero.** Let's see what would this mean to **Entanglement Hamiltonian** defined earlier as

$$K_\Delta := (K_{AB} + K_{BC}) - \eta(K_A + K_C) - (1 - \eta)(K_B + K_{ABC}) \tag{4.22}$$

Since the cross ratio is zero and K_{AB} and K_{BC} are zero since there are only 2 systems. this equation simplifies to

When $\eta = 0$, the equation simplifies to:

$$K\Delta := (K_{AB} + K_{BC}) - 0 - (1 - 0)(K_B + K_{ABC})$$

So, the simplified equation will be

$$K\Delta := (K_{AB} + K_{BC}) - (K_B + K_{ABC})$$

Now, for single interval we know $K_{AB} = K_B$ and $K_{BC} = K_{ABC}$. Substituting these values into the simplified equation, we get:

$$K\Delta := (K_B + K_{ABC}) - (K_B + K_{ABC})$$

This means that all the terms will cancel out, resulting in:

$$K\Delta := 0$$

4.6 Example: Two interval system

We recall that cross-ration defined for 3 intervals was

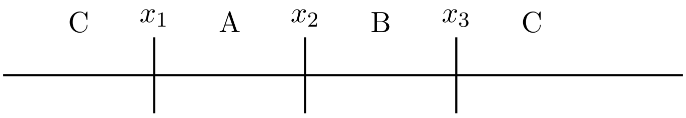

Figure 6: Two intervals (A and B) on an infinite system. The regions outside and between the intervals are labeled as C. Here, $x_{21} := x_2 - x_1$ and $x_{43} := x_4 - x_3$ denote the distances between x_1 and x_2, and x_3 and x_4, respectively.

$$\eta = \frac{(x_2 - x_1)(x_4 - x_3)}{(x_3 - x_1)(x_4 - x_2)} \tag{4.23}$$

For 2 interval

$$\eta = 1 \tag{4.24}$$

Given the equation

$$K\Delta := (K_{AB} + K_{BC}) - \eta(K_A + K_C) - (1 - \eta)(K_B + K_{ABC})$$

When $\eta = 1$, the equation simplifies to:

$$K\Delta := (K_{AB} + K_{BC}) - 1(K_A + K_C) - (1 - 1)(K_B + K_{ABC})$$

So, the simplified equation will be

$$K\Delta := (K_{AB} + K_{BC}) - (K_A + K_C)$$

Now, we know for 2 interval $K_{AB} = K_A$ and $K_{BC} = K_C$. Substituting these values into the simplified equation, we get:

$$K\Delta := (K_A + K_C) - (K_A + K_C)$$

This means that all the terms will cancel out, resulting in:

$$K\Delta := 0$$

So we see that $K\Delta := 0$ in both one and two interval cases.

4.7 Three interval system

Let A, B, and C be three consecutive intervals

We know K_Δ as

$$K_\Delta := (K_{AB} + K_{BC}) - \eta(K_A + K_C) - (1 - \eta)(K_B + K_{ABC}) \tag{4.25}$$

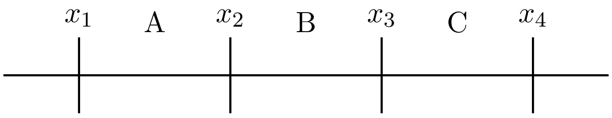

Figure 7: Three consecutive intervals on an infinite system with the corresponding cross ratio η [9] (shown below). Here, $x_{ij} := x_j - x_i$ denotes the distance between x_i and x_j .

And we know that K for conformal ground state is defined as [4]

$$K[x_1, x_2] = 2\pi \int_{x_1}^{x_2} dx \frac{(x - x_1)(x_2 - x)}{x_2 - x_1} T_{00}(x) + \text{const} \tag{4.26}$$

From figure above we see $A = [x_1, x_2]$, $B = [x_2, x_3]$, and $C = [x_3, x_4]$ are the three consecutive intervals. For convenience, the paper defines the function

$$f_{x_0,x_{00}}(x) = \frac{(x - x_0)(x_{00} - x)}{x_{00} - x_0} 1_{[x_0,x_{00}]} \tag{4.27}$$

where $1_{[x_0,x_{00}]}$ is the indicator function. Hence K_Δ written as

$$= \int_{-\infty}^{\infty} dx \left((f_{AB} + f_{BC}) - \eta(f_A + f_C) - (1 - \eta)(f_B + f_{ABC}) \right) T_{00}(x) + \text{const.} \tag{4.28}$$

K_Δ will be $\propto I$ if

$$(f_{AB} + f_{BC}) - \eta(f_A + f_C) - (1 - \eta)(f_B + f_{ABC}) = 0 \tag{4.29}$$

Since f is an indicator function we can evaluate this for each interval separately.

Interval A

$$(f_{AB} + f_{BC}) - \eta(f_A + f_C) - (1 - \eta)(f_B + f_{ABC}) = 0 \tag{4.30}$$

Simplifies too

$$(f_{AB}) - \eta(f_A) - (1 - \eta)(f_{ABC}) = 0 \tag{4.31}$$

Interval B

$$(f_{AB} + f_{BC}) - \eta(f_A + f_C) - (1 - \eta)(f_B + f_{ABC}) = 0 \tag{4.32}$$

Simplifies too

$$(f_{AB} + f_{BC}) - (1 - \eta)(f_B + f_{ABC}) = 0 \tag{4.33}$$

Interval C

$$(f_{AB} + f_{BC}) - \eta(f_A + f_C) - (1 - \eta)(f_B + f_{ABC}) = 0 \tag{4.34}$$

Simplifies too

$$(f_{BC}) - \eta(f_C) - (1 - \eta)(f_{ABC}) = 0 \tag{4.35}$$

Therefore for

$$(f_{AB} + f_{BC}) - \eta(f_A + f_C) - (1 - \eta)(f_B + f_{ABC}) = 0 \tag{4.36}$$

It has to satisfy these 3 equations independently for each interval

$$\boxed{\begin{gathered}(f_{AB}) - \eta(f_A) - (1 - \eta)(f_{ABC}) = 0 \\ (f_{AB} + f_{BC}) - (1 - \eta)(f_B + f_{ABC}) = 0 \\ (f_{BC}) - \eta(f_C) - (1 - \eta)(f_{ABC}) = 0\end{gathered}} \tag{4.37}$$

4.8 Justification for the formula used for K_Δ

One can ask where does this formula comes from. What's the reason behind formulating Entanglement Hamiltonian this way? The paper does not give any reasoning or justification for their formulation. We provide the justification for the formula used.

$$K_\Delta := (K_{AB} + K_{BC}) - \eta(K_A + K_C) - (1 - \eta)(K_B + K_{ABC}) \tag{4.38}$$

As before we place K with indicator function f. Then the condition for $K_\Delta \propto I$ is

$$(f_{AB} + f_{BC}) - \eta(f_A + f_C) - (1 - \eta)(f_B + f_{ABC}) = 0 \tag{4.39}$$

We recall what these f are for each interval

$$f_A = \frac{(x - x1)(x2 - x)}{x2 - x1} \tag{4.40}$$

$$f_B = \frac{(x - x2)(x3 - x)}{x3 - x2}, \tag{4.41}$$

$$f_C = \frac{(x - x3)(x4 - x)}{x4 - x3} \tag{4.42}$$

$$f_{AB} = \frac{(x - x1)(x3 - x)}{x3 - x1} \tag{4.43}$$

$$f_{BC} = \frac{(x - x2)(x4 - x)}{x4 - x2} \tag{4.44}$$

$$f_{ABC} = \frac{(x - x1)(x4 - x)}{x4 - x1} \tag{4.45}$$

Function f is a simple quadratic equation. f for the interval $[0, 2]$ the graph looks as follows

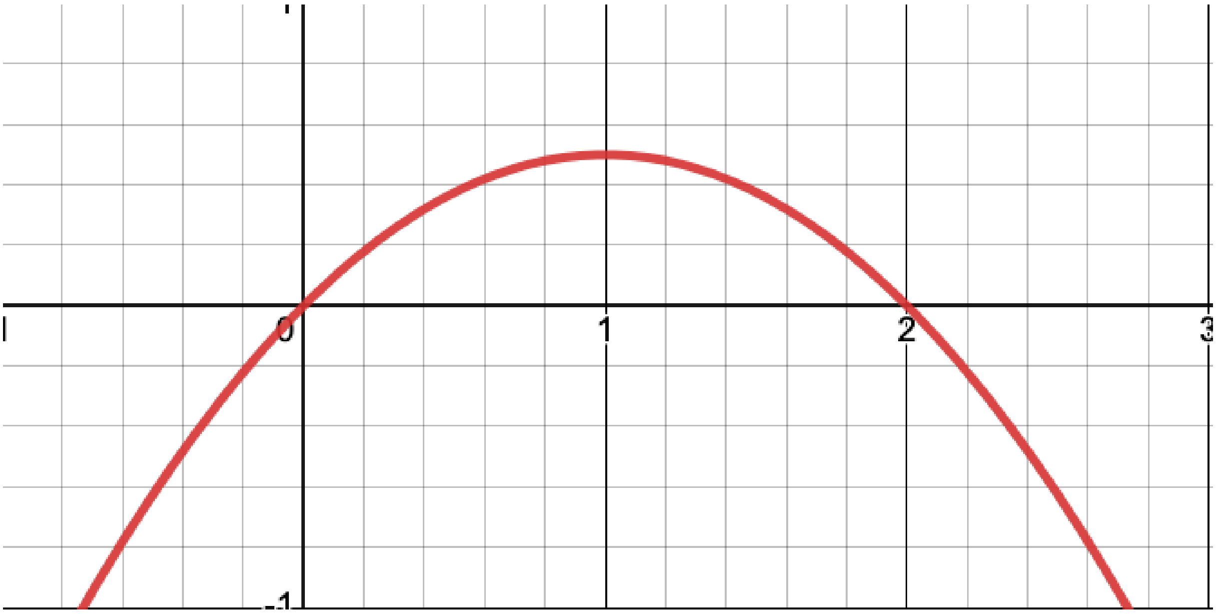

Figure 8: f for the interval $[0, 2]$

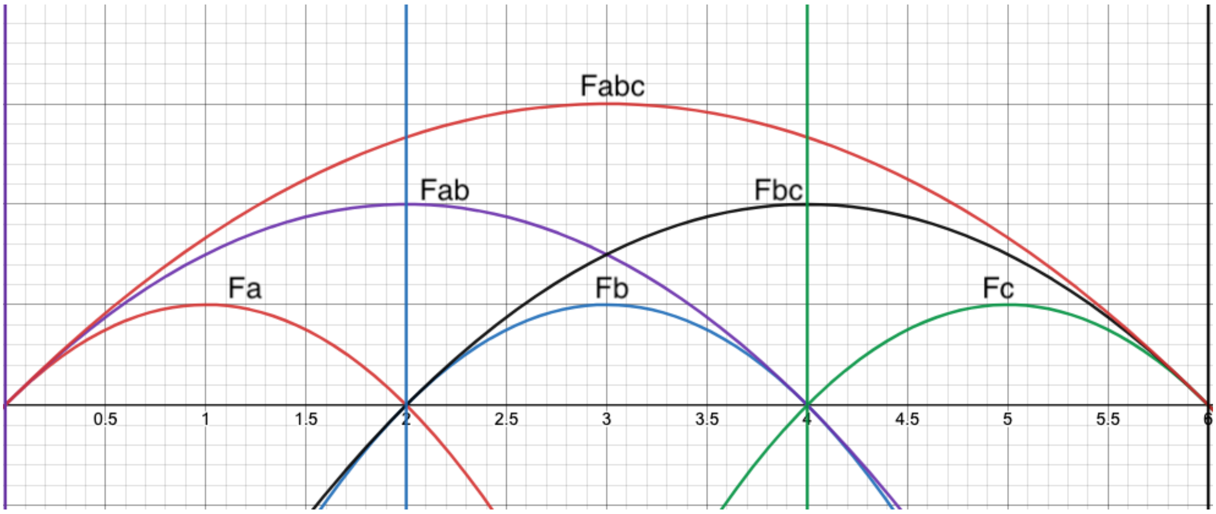

Figure 9: All the f plotted takes the shape above for the interval $A = [0, 2]$, $B = [2, 4]$, and $C = [4, 6]$

All the f above it takes the following shape for the interval $A = [0, 2]$, $B = [2, 4]$, and $C = [4, 6]$ as shown in 9

Since for each interval, all the f has to vanish we get these 3 general equations

$$\alpha f_A + \beta f_{AB} + \gamma f_{ABC} = 0 \tag{4.46}$$

$$\delta f_B + \beta (f_{AB} + f_{BC}) + \gamma f_{ABC} = 0 \tag{4.47}$$

$$\alpha f_C + \beta f_{BC} + \gamma f_{ABC} = 0 \tag{4.48}$$

Where $\alpha, \beta, \gamma$and, δ are unknown constants. We used the same scaling factor (constants) for f_A and f_C and for f_{AB} and f_{BC} because of the symmetry of the equation also illustrated in the graph above. But we have used a different scaling factor for f_B as it does not have similar symmetry.

This leads us to the above-mentioned three equations

$$\alpha f_A + \beta f_{AB} + \gamma f_{ABC} = 0 \tag{4.49}$$

$$\delta f_B + \beta(f_{AB} + f_{BC}) + \gamma f_{ABC} = 0 \tag{4.50}$$

$$\alpha f_C + \beta f_{BC} + \gamma f_{ABC} = 0 \tag{4.51}$$

Solving this we get

$$\beta = \frac{(x1 - x3)(x4 - x2)}{(x1 - x2)(x3 - x4)}\alpha \tag{4.52}$$

$$\gamma = \frac{(x2 - x3)(x1 - x4)}{(x2 - x1)(x4 - x3)}\alpha \tag{4.53}$$

$$\delta = \frac{(x2 - x3)(x1 - x4)}{(x1 - x2)(x3 - x4)}\alpha \tag{4.54}$$

Observations

- All the constants are scaled to a factor of α. Therefore alpha can be ignored as it factorises.
- All the constants are some multiple of η cross ratio defined as before

$$\eta = \frac{(x_2 - x_1)(x_4 - x_3)}{(x_3 - x_1)(x_4 - x_2)} \tag{4.55}$$

- $\gamma = \delta$

Therefore we show that for any linear combinations the conjectured quantity

$$K_\Delta := (K_{AB} + K_{BC}) - \eta(K_A + K_C) - (1 - \eta)(K_B + K_{ABC}) \tag{4.56}$$

is the only quantity that satisfies the condition $K_\Delta \propto I$

4.9 Critical Points of an entropy function

We have seen that $K_\Delta \propto I$ for a 1+1D ground state CFT. Now we show that the states that satisfy this condition is a critical point of an entropy function.

The relation between Entanglement Hamiltonian and Entanglement Entropy is given by

$$\langle\psi|K_A|\psi\rangle = S_A \tag{4.57}$$

For all for all $|\psi\rangle$ satisfying $\langle d\psi|\psi\rangle + \langle\psi|d\psi\rangle = 0$ we have

$$\boxed{dS_\Delta = \langle d\psi|K_\Delta|\psi\rangle + \langle\psi|K_\Delta|d\psi\rangle = 0} \tag{4.58}$$

Let α be a proportionality constant then we have

$$K_\Delta = \alpha I \tag{4.59}$$

We get

$$\begin{aligned} dS_\Delta &= \langle d\psi|K_\Delta|\psi\rangle + \langle\psi|K_\Delta|d\psi\rangle \\ &= \alpha\left(\langle d\psi|\psi\rangle + \langle\psi|d\psi\rangle\right) \\ &= \alpha\cdot 0 \\ &= 0 \end{aligned} \tag{4.60}$$

Hence we have shown that

1. Ground state of 1+1D CFT satisfy $K_\Delta \propto I$
2. States that satisfy $K_\Delta \propto I$ are critical points of entropy function
3. **Therefore Ground of state 1+1D CFT are critical points of the CFT!**

5 Beyond Ground States

An interesting question that arises from the above discussion is can K_Δ be used to identify 1+1D CFT ground states?

The paper [9] does not provide any discussion on this question. But we proceed to explore this question as it is a very interesting one and if true will be very useful in the determination of 1+1D CFT ground states.

Although we know the condition $K_\Delta \propto I$ is satisfied by 1+1D CFT ground states the question is - is it only satisfied by ground states or are the some excited states that satisfy this condition?

The Entanglement Hamiltonian K_ρ of a density matrix ρ which has the form $\rho = \rho_0 + \delta\rho$, where ρ_0 is a reference density matrix whose modular Hamiltonian$K_0 = -\log\rho_0$ is given by [12]

$$K_\rho = K_0 + \sum_{n=1}^{\infty}(-1)^n \int_{-\infty}^{\infty} ds_1 \dots ds_n K_n(s_1,\dots,s_n) \prod_{k=1}^{n}(e^{-(\frac{is_k}{2\pi}+\frac{1}{2})K_0}\delta\rho e^{(\frac{is_k}{2\pi}-\frac{1}{2})K_0}, \tag{5.1}$$

Here:

- K_ρ is the modular Hamiltonian of the excited state.
- K_0 is the modular Hamiltonian of the reference density matrix ρ_0.
- $\delta\rho$ is the deviation of the density matrix ρ from ρ_0.
- $K_n(s_1,\dots,s_n)$ is a kernel function

- The integrals and summations are operations involving the modular flow generated by the reference density matrix ρ_0.

The kernel given as [12]

$$K_n(s_1,\ldots,s_n) = \frac{(2\pi)^2}{(4\pi)^{n+1}} \frac{i^{n-1}}{\cosh\left(\frac{s_1}{2}\right)\cosh\left(\frac{s_n}{2}\right)} Q_n \prod_{k=2}^{n} \frac{1}{\sinh\left(\frac{s_k-s_{k-1}}{2}\right)} \tag{5.2}$$

In ground state $\delta\rho = 0$ which gives us back the equation we used before

$$K_0 = 2\pi \int_A x T_{00}(x) dx \tag{5.3}$$

The question we want to answer is when K is defined as K_ρ for excited states does this still satisfy the condition $K_\Delta \propto I$?

Unfortunately, this depends on the nature of the excitation and the kernel function. Hence we don't make any comments future on this and only mention some interesting points for further study in conclusion.

6 Conclusion and Discussion

6.1 Conclusion

In this report, we have described the necessary background in Conformal Field Theory specifically for 1+1D. In Quantum Information we discussed Entanglement Entropy, Entanglement Hamiltonian for pure and mixed states. Later we described the ground state Entanglement Hamiltonian and Entanglement Entropy for 1+1D CFT.

Then we provided a review of the main claims of the paper [9]. We explain the claims made by the paper in detail with examples. We also justify the conjectured formula used for Entanglement Hamiltonian.

In the end, we also go beyond the reviewed paper and look at the Entanglement Hamiltonian for excited states [12] and show that the ground state Entanglement Hamiltonian can be obtained from the excited state equation.

6.2 Discussion and future study

There were parts of the paper [9] that we found raised more questions than answered. It is not clear that in the conjectured Entanglement Hamiltonian as shown below [9]

$$\boxed{K_\Delta := (K_{AB} + K_{BC}) - \eta(K_A + K_C) - (1-\eta)(K_B + K_{ABC})} \tag{6.1}$$

How can the Entanglement Hamiltonian of different Hilbert spaces be added? In the density matrix formalism, they are simply matrices of different dimensions. The paper [9] does not provide any reasoning or justification for the same.

How did the authors arrive at the equation? Again there is no reference, reasoning, or proof provided.

The paper shows the condition $K_\Delta \propto I$ is indeed satisfied by 1+1D CFT ground states. Whether the converse is true is unclear. If K_Δ defined this way can be used to identify ground states of CFT is unclear from the discussion and needs further investigation.

Are the results shown only true for 1+1D Ground states? Is there is way to define Entanglement Hamiltonian for d=3 that would also satisfy the condition $K_\Delta \propto I$?

Pure states that are not maximally entangled do not satisfy the condition $K_\Delta \propto I$. This forces the condition that all 1+1D CFT ground states have to be maximally mixed states. Is this necessary? Is $K_\Delta \propto I$ too much to ask for?

Is there a different non-linear combination of Entanglement Hamiltonian that would also give similar results?

In order for this formula to be used as an information-theoretic criterion for conformal field theories more work needs to be done.

Personally, I would like to investigate if the condition holds for excited states (the last part of our discussion) and take it further.

References

[1] Pasquale Calabrese and John Cardy. Entanglement entropy and conformal field theory. *Journal of physics a: mathematical and theoretical*, 42(50):504005, 2009.

[2] John Cardy. Boundary conformal field theory. *arXiv preprint hep-th/0411189*, 2004.

[3] John Cardy. Conformal field theory and statistical mechanics. *Exact Methods in Lowdimensional Statistical Physics and Quantum Computing (Oxford University Press, Oxford, 2010)*, pages 65–98, 2010.

[4] John Cardy and Erik Tonni. Entanglement hamiltonians in two-dimensional conformal field theory. *Journal of Statistical Mechanics: Theory and Experiment*, 2016(12):123103, dec 2016.

[5] Marcello Dalmonte, Viktor Eisler, Marco Falconi, and Benoî t Vermersch. Entanglement hamiltonians: From field theory to lattice models and experiments. *Annalen der Physik*, 534(11):2200064, aug 2022.

[6] TH Hansson, M Hermanns, SH Simon, and SF Viefers. Quantum hall physics-hierarchies and cft techniques. *arXiv preprint arXiv:1601.01697*, 2016.

[7] Stephen W Hawking. The information paradox for black holes. *arXiv preprint arXiv:1509.01147*, 2015.

[8] Rolf Landauer. Information is a physical entity. *Physica A: Statistical Mechanics and its applications*, 263(1-4):63–67, 1999.

[9] Ting-Chun Lin and John McGreevy. Conformal field theory ground states as critical points of an entropy function. *arXiv preprint arXiv:2303.05444*, 2023.

[10] Michael A Nielsen and Isaac L Chuang. Quantum computation and quantum information. *Phys. Today*, 54(2):60, 2001.

[11] Blumenhagen Ralph and E Plauschinn. Introduction to conformal field theory: with applications to string theory. *Lecture Notes in Physics. Berlin, Heidelberg: Springer*, 2009.

[12] Gábor Sárosi and Tomonori Ugajin. Modular hamiltonians of excited states, ope blocks and emergent bulk fields. *Journal of High Energy Physics*, 2018(1):1–36, 2018.

[13] Claude Elwood Shannon. A mathematical theory of communication. *The Bell system technical journal*, 27(3):379–423, 1948.

[14] Leonard Susskind. The world as a hologram. *Journal of Mathematical Physics*, 36(11):6377–6396, 1995.

[15] Stephen Blundell Tom Lancaster. *Quantum Field Theory for the Gifted Amateur.* OUP Oxford, Great Clarendon Street, Oxford, OX2 6DP, 2014.